Differential Equations – Revision Notes

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January 18, 2004

Abstract

These quick refresher notes will probably only be useful for those with a univsersity level maths. They were written primarily for my own benefit and so may appear sketchy. They are loosly based on Jordan & Smith and Boas textbooks.

1 Definitions

1.1 Independant / Dependant Variables

With $\frac{dy}{dx}$, x is the *independant variable* as it can take any value, whereas y (a function of x,) is dependant on x and so is the *dependant variable*.

1.2 Differential Equation Definition

A differential equation is an equation that contains at least one *differential* (a function differentiated i.e. $\frac{dy}{dx}$ or $\frac{d^2\psi}{dt^2}$). A solution is a function which, when substituted for the dependant variable give an *identity* (a relationship that is true for all values of the independant variable(s)).

1.3 Linear Differential Equations

Equations that take the form ...

$$\frac{d^n y}{dx^n} + \ldots + g(x)\frac{d^2 y}{dx^2} + h(x)\frac{dy}{dx} + i(x)y = f(x)$$

are linear equations. The order of the linear equations is the order of the highest differential.

1.4 Forcing term

In the above linear equation $f(x)$ is the *forcing term.* This represents the input of the system wheras the solution represents the output. When this is zero the equation is said to be *unforced* or *homogenous.*

1.5 Boundary Conditions

There are often many solutions to a given equation. However they can all be obtained from the *general solution* which features n arbitrary constants where n is the order of the equation. To go from this to a specific or *particular solutions*, we often specify *boundary conditions.* i.e. At $x = 0, y = 0$. In general, we need as many conditions as there are arbitrary constants to get a particular solution.

2 Solutions to Unforced Linear Equations with Constant Coefficients

2.1 First Order

$$\frac{dy}{dx} + cy = 0 \qquad \text{Solution:} y = Ae^{-ct}$$

c is a constant and A is an arbitrary constant.

2.2 Second Order

$$\frac{d^2x}{dt^2} + b\frac{dx}{dt} + cx = 0$$

Try solns. of the form $x(t) = e^{mt}$ then sub in the above to get ...

$$m^2e^{mt} + bme^{mt} + ce^{mt} = e^{mt}(m^2 + bm + c) = 0$$

Solve the above quadratic (known as *characteristic equation*,) to get m_1 and m_2. This gives ...

$$x(t) = e^{m_1t} \text{ and } x(t) = e^{m_2t}$$

This is know as the *basis.* By superposition theorem give family of solns.

$$x(t) = Ae^{m_1t} + Be^{m_2t}$$

Note: each 2dn order eqn. has two linearly independant solns.

2.2.1 If characterstic eqn. only has one soln.

If characteristic eqn end up like $(m+2)^2$ then other soln. is of form ...

$$te^{mt} \text{so general soln. is} \ldots Ae^{mt} + Bte^{mt}$$

2.2.2 If characteristic eqn. has complex solns.

Work out as normal ...

$$x(t) = Ae^{(\alpha+i\beta)t} + Be^{(\alpha-i\beta)t}$$

but can be simplified. i.e.

$$e^{(\alpha+i\beta)t} = e^{\alpha t}e^{i\beta t} = e^{\alpha t}\cos\beta t + ie^{\alpha t}\sin\beta t$$

(by de Moivre). The Real and Imaginary part of the above give a new basis in their own right. The above can be simplified further ...

$$x(t) = e^{\alpha t}\cos\beta t + e^{\alpha t}\sin\beta t = Ce^{\alpha t}\cos(\beta t + \phi)$$

C and ϕ are arbitrary consts. (Think conversion between cartesian and polar co-ordinates).

3 Solutions to Forced Linear Eqns.

The following obtain particular solutions. Particular solutions. are used to obtain general solutions. (explained later).

3.1 Particular solns.

Forcing term is $f(t)$. Finding particular solns. is case of trying an appropriate solution and adjusting parameters until it fits.

3.1.1 First and second order constant coefficient linear equations

As a general rule the following suggestions will work,

Forcing term ($f(t)$)	Solution to try ($x(t)$)
$Ke^{\alpha t}$	$pe^{\alpha t}$
$K\cos\beta t$ or $K\sin\beta t$	$p\cos\beta t + q\sin\beta t$
Polynomial order N	Complete polynomial order N

If $f(t)$ is a sum of the above, then due to the superposition principle we can calculate particular solutions for each seperately then simply sum the solutions.

Sometimes the above suggestion give trivial (null) solutions. The following are some exceptional cases and their solutions.

Equation	Solution to try ($x(t)$)
$\frac{d^2x}{dt^2} + \beta^2 x = a\cos\beta t$	$\frac{a}{2\beta} t \sin\beta t$
$\frac{d^2x}{dt^2} + \beta^2 x = a\sin\beta t$	$-\frac{a}{2\beta} t \cos\beta t$

3.1.2 Using de Moivre

When the forcing term is of form $ae^{\alpha t}\sin\beta t$ or $ae^{\alpha t}\cos\beta t$ solve using forcing term $ae^{(\alpha+i\beta)}$ then take the real or imaginary part of the solution as appropriate.

3.2 General solutions

By clever jiggery-pokery, we obtain general solutions by summing an arbitrary particular solution of the forced equation with the general solution of the equivelant unforced equation, (i.e. where we substitute $f(x)$ for 0).

4 Linear Equations with non-constant coefficients

4.1 First order

Equations of form,

$$\frac{dx}{dt} + g(t)x = f(t)$$

Then solution can be got from,

$$xe^I = \int f(t)e^I + c \qquad \text{where} \qquad I = \int g(x)dx$$

Note: This works for constant coefficients as well.

4.2 Equations with lower order terms missing

An example,

$$g(x)\frac{d^2x}{dt^2} + h(t)\frac{dx}{dt} = f(t)$$

There is no term $i(t)x$. Substitute in a new temporary dependant variable (say, p), such that $p = \frac{dx}{dt}$ and $\frac{dp}{dt} = \frac{d^2x}{dt^2}$ thus converting it into a first order equation. Solve using the above then replace p with $\frac{dx}{dt}$ when you have the solution.

4.3 First order seperable

Equations of form,

$$\frac{dx}{dt} = g(x)h(t)$$

rearrange into the following form,

$$\frac{dx}{g(x)} = h(t)dt$$

then integrate both sides.

4.4 Cauchy/Euler equations

Equations of form,

$$at^2\frac{dx}{dt} + bt\frac{dx}{dt} + cx = f(t)$$

Can be made into a linear equation with constant coefficients by using $t = e^v$. This gives following substitutions,

$$x\frac{dx}{dt} = \frac{dx}{dv} \qquad \text{and} \qquad x^2\frac{d^2x}{dt^2} = \frac{d^2x}{dv^2} - \frac{dx}{dv}$$

5 Non-linear equations

These tend not to crop up much in practice, so more complex techniques are glossed over here. The Principle of Superposition does not apply to non-linear equations, therefore many of the techniques are just ways to reduce a non-linear equation into a linear one.

5.1 Nearly seperable equations

If an equation is of the form,

$$\frac{dx}{dt} = f\left(\frac{x}{t}\right)$$

Make the substitution, $v = x/t$. Note: for left hand side use $x = vt$ to get $dx/dt = xdv/dt + v$, solve as seperable with v as dependant variable and then substitute back.

5.2 Bernoulli equations

For equations of form (non-linear due to x^n term),

$$\frac{dx}{dt} + g(t)x = h(t)x^n$$

We change the variables using $v = x^{1-n} \Rightarrow \frac{dv}{dt} = (1-n)x^{-n}\frac{dx}{dt}$. We do this by first multiplying the above by $(1-n)x^{-n}$. This gives,

$$(1-n)x^{-n}\frac{dx}{dt} + (1-n)g(t)x^{1-n} = (1-n)h(t)$$

Substituting in v gives,

$$\frac{dv}{dt} + (1-n)g(t)v = (1-n)h(t)$$

Solve, then substitute x back in.

5.3 Second order, non

6 Partial Differential Equations

A differential equation with more than one independant variable

6.1 Seperation of variables

Assume that solution is the product of functions of each of the independant variables. i.e. for the wave equation,

$$\frac{\partial y^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial y^2 y}{\partial t^2}$$

Assume that $y(x,t) = X(x)T(t)$. Substitute in, then divide by XT to get,

$$\frac{1}{X}\frac{d^2X}{dx^2} = \frac{1}{v^2}\frac{1}{T}\frac{d^2T}{dt^2}$$

Since the above needs to be true for all x and t (an identity), it must be that each side is equal to a constant. Therefore from this we can extract two second order differential equations of one independant variable, as below. (We set the constant to be $-k^2$ to simplify the solution.

$$\frac{1}{X}\frac{d^2X}{dx^2} = -k^2 \qquad \text{and} \qquad -\frac{1}{v^2}\frac{1}{T}\frac{d^2T}{dt^2} = -k^2$$

$$\Rightarrow \frac{d^2X}{dx^2} = -k^2X \qquad \text{and} \qquad -\frac{d^2T}{dt^2} = -k^2v^2T$$

Solve these then multiply solutions of X and T to get y.